Corrigendum to "A Fixed Point Theorem for Measurable Selection Valued Correspondences Induced by Upper Caratheodory Correspondences"

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1 Introduction

In our paper, "A Fixed Point Theorem for Measurable Selection Valued Correspondences Induced by Upper Caratheodory Correspondences," (Fu and Page, 2023, 2022b) we argue that any measurable selection valued correspondence, $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$, induced by the composition of an m-tuple of real-valued Caratheodory functions,

$$(\omega, y, x) \longrightarrow U(\omega, v, x) := (U_1(\omega, v_1, x), \ldots, U_m(\omega, v_m, x)), \tag{1}$$

with an upper Caratheodory (uC) correspondence,

$$(\omega, v) \longrightarrow \mathcal{N}(\omega, v), \tag{2}$$

i.e.,

$$\left.\begin{aligned} (\omega, y, x) \longrightarrow \mathcal{P}(\omega, v) := U(\omega, v, \mathcal{N}(\omega, v)) \\ := \{(U_1(\omega, v_1, x), \ldots, U_m(\omega, v_m, x)) : x \in \mathcal{N}(\omega, v)\} \end{aligned}\right\} \tag{3}$$

has fixed points if the underlying uC correspondence, $\mathcal{N}(\cdot, \cdot)$, in the composition, contains a continuum-valued uC sub-correspondence, $\eta(\cdot, \cdot)$.[1] *This statement is incorrect.* The presence of a continuum-valued uC sub-correspondence is not enough. In order to establish that the induced selection correspondence, $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$, has fixed points, the uC composition correspondence must induce a *contractibly-valued* selection sub-correspondence. Here we will show that in many problems involving Caratheodory compositions, as in (3), especially those arising in discounted stochastic games where the m-tuple of parameters is given by an m-tuple of essentially bounded measurable functions,

$$v = (v_1, \ldots, v_m) \in \mathcal{L}^{\infty}_{Y_1} \times \cdots \times \mathcal{L}^{\infty}_{Y_m} := \mathcal{L}^{\infty}_{Y},$$

with $-M \leq v_d(\omega') \leq M$ a.e. $[\mu]$, the underlying structure of the problem is such that the induced selection correspondence, $v \longrightarrow \mathcal{S}^{\infty}(\mathcal{P}(\cdot, v))$, naturally contains a contractibly-valued sub-correspondence. The key property that ensures that this is the case, is the *K-limit property.* Stated informally, *a selection sub-correspondence has the K-limit property if its graph contains all of its K-limits* (i.e., its Komlos limits - Komlos, 1967). Under precisely the same assumptions as in our prior paper, but with the following strengthening of those assumptions - (1) that the probability measure governing the state space is *nonatomic*, and (2) that the m-tuple real-valued Caratheodory functions,

$$(\omega, v, x) \longrightarrow U(\omega, v, x) := (U_1(\omega, v_1, x), \ldots, U_m(\omega, v_m, x)),$$

in (3), is such that each player's payoff function, $(\omega, v_d, x) \longrightarrow U_d(\omega, v_d, x)$, is *affine* in v_d - we will give a correct statement and proof of our fixed point result. We will refer to the functions, (1), as *affinely parameterized Caratheodory functions* (i.e., measurable in $\omega \in \Omega$, jointly continuous in $(v_d, x) \in \mathcal{L}^{\infty}_{Y_d} \times X$, and affine in the

[1] Here $U(\cdot, \cdot, \cdot)$ is measurable in ω and jointly continuous in (v, x) and $\mathcal{N}(\cdot, \cdot)$ is jointly measurable in (ω, v) and upper semicontinuous in v.

parameter, v_d). Under our strengthened set of assumptions (all listed below), we take as our starting point a continuum-valued, uC composition correspondence,

$$(\omega, v) \longrightarrow p(\omega, v) := U(\omega, v, \eta(\omega, v))$$

$$:= \{(U_1(\omega, v_1, x), \ldots, U_m(\omega, v_m, x)) : x \in \eta(\omega, v)\},$$

induced by a continuum-valued uC sub-correspondence, $\eta(\cdot, \cdot)$, and using a result due to Page (1991) we show that the induced selection sub-correspondence, $\mathcal{S}^{\infty}(p_{(\cdot)})$, has the K-limit property.[2] Given the near equivalence of K-convergence and weak star convergence, as a consequence of $\mathcal{S}^{\infty}(p_{(\cdot)})$ having the K-limit property, we are able to show that any such selection correspondence, $\mathcal{S}^{\infty}(p_{(\cdot)})$, is *upper semicontinuous and takes contractible values* (with respect to the weak star topologies) - implying that $\mathcal{S}^{\infty}(p_{(\cdot)})$ is *approximable* (with respect to the weak star topologies) and *has fixed points* (i.e., $v^* \in \mathcal{L}_Y^{\infty}$ such that $v^* \in \mathcal{S}^{\infty}(p_{v^*})$).

The sequence of arguments and implications described above is precisely the line of reasoning which must be brought to bear upon the fixed point problem that arises in connection with the Nash payoff selection correspondence belonging to a ***nonatomic***, ***convex discounted stochastic game*** *(DSG). In a nonatomic, convex DSG, the Nash payoff selection correspondence arises from the composition of the upper Caratheodory Nash correspondence with the m-tuple of affinely parameterized, uniformly bounded, real-valued, Caratheodory player payoff functions - inducing, therefore, an approximable Nash payoff selection correspondence having fixed points.* It then follows from Blackwell's Theorem (1965) extended to $DSGs$, that all nonatomic, convex $DSGs$ have stationary Markov perfect equilibria ($SMPE$). We note that the presence of an upper semicontinuous Nash payoff selection sub-correspondence taking *contractible* values is critical to the existence of stationary Markov perfect equilibria. Upper semicontinuity and contractibility guarantee the existence of one-shot Nash equilibria and *rule out circular Nash payoffs* (i.e., Nash payoffs homeomorphic to the unit circle). Thus, upper semicontinuity and contractibility eliminate the key building block for constructing counterexamples to the existence of $SMPE$ (see Levy, 2013, and Levy and McLennan, 2015). Also, we show by example that the assumption that players' state-contingent feasible action sets are convex is critical to establishing that the Nash payoff selection correspondence contains a sub-correspondence that is upper semicontinuous with respect to the weak star topologies.

Now to the details.

2 Primitives and Assumptions

We will maintain the following list of (strengthened) assumptions throughout, labeled [A-1] - a list essentially the same as in our original paper paper. We will present our

[2] Fu and Page (2022a) have shown that all one-shot games underlying an uncountable state, convex discounted stochastic games have Nash correspondences containing a continuum-valued, minimal uC Nash sub-correspondence taking minimally essential Nash values (essential in the sense of Fort, 1950).

fixed point result in a game theoretic setting, but the result has applications outside of game theory.

(1) $D =$ the set of players, consisting of m players indexed by $d = 1, 2, \ldots, m$ and each having discount rate given by $\beta_d \in (0, 1)$.
(2) (Ω, B_Ω, μ), the state space where Ω is a complete separable metric spaces with metric ρ_Ω, equipped with the Borel σ-field, B_Ω, upon which is defined a ***nonatomic*** *probability measure, μ, often referred to as the dominating probability measure.*[3]
(3) $Y := Y_1 \times \cdots \times Y_m$, the space of players' payoff profiles, $y := (y_1, \ldots, y_m)$, such that for each player d, $Y_d := [-M, M]$, $M > 0$, equipped with the absolute value metric, $\rho_{Y_d}(y_d, y'_d) := |y_d - y'_d|$ and Y is equipped with the sum metric, $\rho_Y := \sum_d \rho_{Y_d}$.
(4) $X := X_1 \times \cdots \times X_m := \prod_d X_d \subset E := \prod_d E_d$, the space of player action profiles, $x := (x_1, \ldots, x_m)$, such that for each player d, X_d is a ***convex****, compact metrizable subset of a locally convex Hausdorff topological vector space E_d and is equipped with a metric, ρ_{X_d}, compatible with the locally convex topology inherited from E_d, and X is equipped with the sum metric, $\rho_X := \sum_d \rho_{X_d}$.*
(5) $\omega \longrightarrow \Phi_d(\omega)$, is player d's measurable action constraint correspondence, defined on Ω taking nonempty, ***convex****, ρ_{X_d}-closed (and hence compact) values in X_d.*
(6) $\omega \longrightarrow \Phi(\omega) := \Phi_1(\omega) \times \cdots \times \Phi_m(\omega)$, players' measurable action profile constraint correspondence, defined on Ω taking nonempty, convex, and ρ_X-closed (hence compact) values in X.
(7) $\mathcal{S}^\infty(\Phi_d(\cdot))$, the set of all μ-equivalence classes of (B_Ω, B_{X_d})-measurable functions (selections), $x_d(\cdot)$, defined on Ω such that in $x_d(\omega) \in \Phi_d(\omega)$ a.e. $[\mu]$, and

$$\mathcal{S}^\infty(\Phi(\cdot)) = \mathcal{S}^\infty(\Phi_1(\cdot)) \times \cdots \times \mathcal{S}^\infty(\Phi_m(\cdot)) \tag{4}$$

the set of all μ-equivalence classes of measurable profiles (selection profiles), $x(\cdot) = (x_1(\cdot), \ldots, x_m(\cdot))$, defined on Ω such that

$$x(\omega) \in \Phi(\omega) := \Phi_1(\omega) \times \cdots \times \Phi_m(\omega) \text{ a.e. } [\mu].$$

(8) $\mathcal{L}^\infty_{Y_d}$, the Banach space of all μ-equivalence classes of measurable (value) functions, $v_d(\cdot)$, defined on Ω with values in Y_d a.e. $[\mu]$, equipped with metric $\rho_{w^*_d}$ compatible with the weak star topology inherited from $\mathcal{L}^\infty_R$.
(9) $\mathcal{L}^\infty_Y := \mathcal{L}^\infty_{Y_1} \times \cdots \times \mathcal{L}^\infty_{Y_m} \subset \mathcal{L}^\infty_{R^m}$, the Banach space of all μ-equivalence classes of measurable (value) function profiles, $v(\cdot) := (v_1(\cdot), \ldots, v_m(\cdot))$, defined on Ω with values in Y a.e. $[\mu]$, equipped with the sum metric $\rho_{w^*} := \sum_d \rho_{w^*_d}$ compatible with the weak star product topology inherited from $\mathcal{L}^\infty_{R^m}$.

[3] We will refer to the nonatomic probability measure, μ, as the dominating probability measure. $E \subset \Omega$ is an atom of Ω relative to μ if the following implication holds: if $\mu(E) > 0$, then $H \subset E$ implies that $\mu(H) = 0$ or $\mu(E - H) = 0$. If Ω contains no atoms relative to μ, then μ is a nonatomic probability measure. Because Ω, is a complete, separable metric space $\mu(\cdot)$ is nonatomic if and only if $\mu(\{\omega\}) = 0$ for all $\omega \in \Omega$ (see Hildenbrand, 1974, pp 44-45). Also, note that the σ-field, B_Ω is countably generated. All the results we present here remain valid if instead we assume that Ω is an abstract set, but one equipped with a countably generated σ-field (Ash, 1972).

(10) $u_d(\cdot,\cdot,\cdot):\Omega\times Y_d\times X\longrightarrow Y_d$*, player* d*'s Caratheodory payoff function, measurable in* $\omega\in\Omega$*, jointly continuous in* $(y_d,x)\in Y_d\times X$*,* ***affine*** *in* $y_d\in Y_d$*, and* ***quasiconcave*** *in* $x_d\in X_d$*.*[4]
(11) $q(\cdot|\cdot,\cdot):\Omega\times X\longrightarrow\Delta(\Omega)$ *is the law of motion defined on* $\Omega\times X$ *taking values in the space of probability measures on* Ω*, having the following properties:* ***(i)*** *each probability measure,* $q(\cdot|\omega,x)$*, in the collection*

$$Q(\Omega\times X):=\{q(\cdot|\omega,x):(\omega,x)\in\Omega\times X\}\tag{5}$$

is absolutely continuous with respect to μ *(denoted* $Q(\Omega\times X)<<\mu$*),* ***(ii)*** *for each* $E\in B_\Omega$*,* $q(E|\cdot,\cdot)$ *is measurable on* $\Omega\times X$*, and* ***(iii)*** *the collection of probability density functions,*

$$H_\mu:=\{h(\cdot|\omega,x):(\omega,x)\in\Omega\times X\}\subset\mathcal{L}^1_R(\Omega),\tag{6}$$

of $q(\cdot|\omega,x)$ *with respect to* μ *is such that for each state* ω*, the function*

$$(x_d,x_{-d})\longrightarrow h(\omega'|\omega,x_d,x_{-d})\tag{7}$$

is continuous in x *and affine in* x_d *a.e.* $[\mu]$ *in* ω'*.*[5] *Sometimes we will assume that* ***(iv)*** *for each* $\omega\in\Omega$*, the collection of densities,* H_μ*, is uniformly* μ*-continuous, i.e., for each* $\varepsilon>0$*, there is a* $\delta>0$ *such that if for* $S\in B_\Omega$*,* $\mu(S)<\delta$*, then* $h(S|\omega,x)=\int_S h(\omega'|\omega,x)d\mu(\omega')<\varepsilon$ *for all* $x\in X$*.*

2.1 Remarks

(1) Under assumptions [A-1](10), for each player d in each state ω, the payoff function (integrand), $(y_d,x)\longrightarrow u_d(\omega,y_d,x)$ is jointly continuous on the compact product set $Y_d\times X$, implying that for each ω, $u_d(\omega,\cdot,\cdot)$ is uniformly continuous on the compact product set, $Y_d\times X$, and as a consequence, for each ω, the affinely y_d-parameterized collection of state ω payoff integrands, $\{u_d(\omega,y_d,\cdot):y_d\in Y_d\}$ is uniformly equicontinuous. Thus if $x^n\underset{\rho_X}{\longrightarrow}x^*$, then for any $\varepsilon>0$ there is an integer, N_ε, such that if $n\geq N_\varepsilon$, then $|u_d(\omega,y_d,x^n)-u_d(\omega,y_d,x^*)|<\varepsilon$ for all $y_d\in Y_d$. From this we can deduce that if $v^n_d\underset{\rho_{w^*_d}}{\longrightarrow}v^*_d$ and $x^n\underset{\rho_X}{\longrightarrow}x^*$ then

$$u_d(\omega,v^n_d(\cdot),x^n)\underset{\rho_{w^*_d}}{\longrightarrow}u_d(\omega,v^*_d(\cdot),x^*).$$

[4] $u_d(\omega,\cdot,x)$ is affine in y_d if for y^0_d and y^1_d if for $\alpha\in[0,1]$,

$$\begin{aligned}&u_d(\omega,\alpha y^0_d+(1-\alpha)y^1_d,x)\\&=\alpha u_d(\omega,y^0_d,x)+(1-\alpha)\alpha u_d(\omega,y^1_d,x).\end{aligned}$$

$u_d(\omega,y_d,\cdot,x_{-d})$ is quasiconcave in x_d if for x^0_d and x^1_d if for $\alpha\in[0,1]$,

$$\begin{aligned}&u_d(\omega,y_d,\alpha x^0_d+(1-\alpha)x^1_d,x_{-d})\\&\geq\alpha u_d(\omega,y_d,x^0_d,x_{-d})+(1-\alpha)\alpha u_d(\omega,y_d,x^1_d,x_{-d}).\end{aligned}$$

[5] $\mathcal{L}^1_R(\Omega)$ is the Banach space of μ-equivalence classes *of* integrable real-valued functions defined on Ω.

To see this, suppose we have a subsequence, $\{u_d(\omega, v_d^{n_k}(\cdot), x^{n_k})\}_k$, such that for some $\varepsilon' > 0$ and player d'

$$\rho_{w^*_{d'}}(u_d(\omega, v_d^{n_k}(\cdot), x^{n_k}), u_d(\omega, v_d^*(\cdot), x^*)) > \varepsilon'. \tag{8}$$

We have for player d' and $g(\cdot)$ an arbitrary element of the dual, $\mathcal{L}^1_R$, that

$$\left.\begin{array}{l} \left|\int_\Omega \left[\left(u_{d'}(\omega, v_{d'}^{n_k}(\omega'), x^{n_k}) - u_{d'}(\omega, v_{d'}^*(\omega'), x^*)\right) \cdot g(\omega')\right] d\mu(\omega')\right| \\ \leq \underbrace{\int_\Omega \left|u_{d'}(\omega, v_{d'}^{n_k}(\omega'), x^{n_k}) - u_{d'}(\omega, v_{d'}^{n_k}(\omega'), x^*)\right| \cdot \left|g(\omega')\right| d\mu(\omega')}_{(A)} \\ + \underbrace{\left|\int_\Omega \left[\left(u_{d'}(\omega, v_{d'}^{n_k}(\omega'), x^*) - u_{d'}(\omega, v_{d'}^*(\omega'), x^*)\right) \cdot g(\omega')\right] d\mu(\omega')\right|}_{(B)}. \end{array}\right\} \tag{9}$$

First for (A) we have by uniform equicontinuity that for k sufficiently large,

$$\left|u_{d'}(\omega, v_{d'}^{n_k}(\omega'), x^{n_k}) - u_{d'}(\omega, v_{d'}^{n_k}(\omega'), x^*)\right| \leq \frac{\varepsilon'}{4m}, \tag{10}$$

for any value of $v_{d'}^{n_k}(\omega')$. Therefore, we have for (A)

$$\left.\begin{array}{l} (A) = \left|\int_\Omega \left[\left(u_{d'}(\omega, v_{d'}^{n_k}(\omega'), x^{n_k}) - u_{d'}(\omega, v_{d'}^{n_k}(\omega'), x^*)\right) \cdot g(\omega')\right] d\mu(\omega')\right| \\ \leq \int_\Omega \left|u_{d'}(\omega, v_{d'}^{n_k}(\omega'), x^{n_k}) - u_{d'}(\omega, v_{d'}^{n_k}(\omega'), x^*)\right| \cdot |g(\omega')| \, d\mu(\omega') \\ \frac{\varepsilon'}{4m} \int_\Omega |g(\omega')| \, d\mu(\omega') = \frac{\varepsilon'}{4m} \|g(\cdot)\|_1 \longrightarrow 0 \text{ as } \varepsilon' \downarrow 0. \end{array}\right\} \tag{11}$$

To see that $(B) \longrightarrow 0$, observe that because $v_d^{n_k} \underset{\rho_{w^*_d}}{\longrightarrow} v_d^*$, we can assume that $v_{d'}^{n_k} \underset{K}{\longrightarrow} v_{d'}^*$, implying via the affinity of $u_{d'}(\omega, \cdot, x^*)$ on $Y_{d'}$ that $u_{d'}(\omega, v_{d'}^{n_k}(\cdot), x^*) \underset{K}{\longrightarrow} u_{d'}(\omega, v_{d'}^*(\cdot), x^*)$, further implying via the near equivalence of K convergence and $\rho_{w^*_d}$-convergence that $u_{d'}(\omega, v_{d'}^{n_k}(\cdot), x^*) \underset{\rho_{w^*_{d'}}}{\longrightarrow} u_{d'}(\omega, v_{d'}^*(\cdot), x^*)$. Hence $(B) \longrightarrow 0$. Together, $(A) \longrightarrow 0$ and $(B) \longrightarrow 0$ contradict (8).

(2) We note that because $h(\omega'|\omega, x) \geq 0$ for all (ω, ω', x) and $\int_\Omega h(\omega'|\omega, x) d\mu(\omega') = 1$ *for all* $(\omega, x) \in \Omega \times X$, the collection of densities, $H_\mu \subset \mathcal{L}^1_R$,is uniformly norm bounded in $\mathcal{L}^1_R$ (i.e., $\|h(\cdot|\omega, x)\|_1 = 1$ for all $(\omega, x) \in \Omega \times X$) and therefore, *the uniform* μ*-continuity of* H_μ (see [A-11](iv) above), implies that $H_\mu \subset \mathcal{L}^1_R$ is uniformly integrable, i.e., *for each* $\varepsilon > 0$*, there is a* $c_\varepsilon > 0$ *such that for each* $c \geq c_\varepsilon$,

$$\int_{\{\omega' : h(\omega'|\omega, x) > c\}} h(\omega'|\omega, x) d\mu(\omega') < \varepsilon \text{ for all } x \in X.$$

Because $H_\mu \subset \mathcal{L}^1_R$ is norm bounded and μ-continuous - and therefore, uniformly integrable - if for any pair of sequences, $\{v_d^n\}_n$ and $\{x^n\}_n$, we have $v_d^n \underset{\rho_{w^*_d}}{\longrightarrow} v_d^*$ and

$x^n \underset{\rho_X}{\longrightarrow} x^*$, so that by assumptions [A-1](11)(i)-(iv), $h(\omega'|\omega, x^n) \underset{R}{\longrightarrow} h(\omega'|\omega, x^*)$ a.e. $[\mu]$ in ω', implies that

$$\|h(\cdot|\omega, x^n) - h(\cdot|\omega, x^*)\|_1 \longrightarrow 0. \tag{12}$$

Moreover, under assumptions [A-1](10) and [A-1](11)(i)-(iv), if $v_d^n \underset{\rho_{w_d^*}}{\longrightarrow} v_d^*$ and $x^n \underset{\rho_X}{\longrightarrow} x^*$ then by Remark (1) above we have that the sequence, $\{u_d(\omega, v_d^n(\cdot), x^n)\}_n \subset \mathcal{L}_{Y_d}^\infty \subset \mathcal{L}_R^\infty$, is such that

$$u_d(\omega, v_d^n(\cdot), x^n) \underset{\rho_{w_d^*}}{\longrightarrow} u_d(\omega, v_d^*(\cdot), x^*) \in \mathcal{L}_{Y_d}^\infty. \tag{13}$$

Thus, (12) and (13) imply that $u_d(\omega, v_d^n(\cdot), x^n)$ converges weak star to $u_d(\omega, v_d^*(\cdot), x^*)$ in $\mathcal{L}_{Y_d}^\infty$ where $\{h(\cdot|\omega, x^n)\}_n$ in $\mathcal{L}_R^1$ the separable norm dual of $\mathcal{L}_R^\infty$ converges in $\mathcal{L}_R^1$-norm to $h(\cdot|\omega, x^*) \in \mathcal{L}_R^1$. Therefore, we have that

$$\textstyle\int_\Omega u_d(\omega, v_d^n(\omega'), x^n) h(\omega'|\omega, x^n) d\mu(\omega') \underset{R}{\longrightarrow} \int_\Omega u_d(\omega, v_d^*(\omega'), x^*) h(\omega'|\omega, x^*) d\mu(\omega').$$

We note that if $u_d(\omega, v_d^n(\cdot), x^n) = v_d^n(\cdot)$ for all n, d and ω, then under assumptions [A-1](11)(i)-(iii) - without [A-1](11)(iv) - if $v_d^n \underset{\rho_{w_d^*}}{\longrightarrow} v_d^*$ and $x^n \underset{\rho_X}{\longrightarrow} x^*$, then

$$\textstyle\int_\Omega v_d^n(\omega') h(\omega'|\omega, x^n) d\mu(\omega') \underset{R}{\longrightarrow} \int_\Omega v_d^*(\omega') h(\omega'|\omega, x^*) d\mu(\omega').$$

This is because

$$\left|\textstyle\int_\Omega v_d^n(\omega') h(\omega'|\omega, x^n) d\mu(\omega') - \int_\Omega v_d^*(\omega') h(\omega'|\omega, x^*) d\mu(\omega')\right|$$

$$\leq \underbrace{\left|\int_\Omega v_d^n(\omega') h(\omega'|\omega, x^n) d\mu(\omega') - \int_\Omega v_d^n(\omega') h(\omega'|\omega, x^*) d\mu(\omega')\right|}_{(a)}$$

$$+ \underbrace{\left|\int_\Omega v_d^n(\omega') h(\omega'|\omega, x^*) d\mu(\omega') - \int_\Omega v_d^*(\omega') h(\omega'|\omega, x^*) d\mu(\omega')\right|}_{(b)},$$

and under assumptions [A-1](8) and (11)(i)-(iii),

$$(a) \leq M \textstyle\int_\Omega |h(\omega'|\omega, x^n) - h(\omega'|\omega, x^*)|\, d\mu(\omega') \longrightarrow 0,$$

and because $v_d^n \underset{\rho_{w_d^*}}{\longrightarrow} v_d^*$ and $h(\cdot|\omega, x^*) \in \mathcal{L}_R^1$ the separable norm dual of $\mathcal{L}_R^\infty$,

$$(b) = \left|\textstyle\int_\Omega (v_d^n(\omega') h(\omega'|\omega, x^*) d\mu(\omega') - \int_\Omega v_d^*(\omega')) h(\omega'|\omega, x^*) d\mu(\omega')\right| \longrightarrow 0.$$

3 One-Shot Games, Nash equilibria, and the Three Nash Correspondences

3.1 One-Shot Games

In a strategic form game satisfying assumptions [A-1] above, each player's expected payoff function depends on the current state, $\omega \in \Omega$, the player's coming state prices, as represented by an essentially bounded measurable value function, $\omega' \longrightarrow v_d(\omega')$, of the coming state $\omega' \in \Omega$ (i.e., the player's value function $v_d \in \mathcal{L}^{\infty}_{Y_d}$), and the profile (or m-tuple) of feasible actions,

$$(x_d, x_{-d}) \in \Phi_d(\omega) \times \Phi_{-d}(\omega) \subset X_d \times X_{-d} := X \tag{14}$$

chosen by all the players (i.e., player $d's$ feasible action choice, $x_d \in \Phi_d(\omega) \subset X_d$ by player d and feasible action choices, $x_{-d} \in \Phi_{-d}(\omega) \subset X_{-d}$ by the other players). Thus, in current state ω, if player $d's$ value function is $\omega' \longrightarrow v_d(\omega')$, then player $d's$ expected payoff is given by,

$$U_d(\omega, v_d, x) := \int_{\Omega} u_d(\omega, v_d(\omega'), x) h(\omega'|\omega, x) d\mu(\omega'), \tag{15}$$

if players choose action profile $x \in \Phi(\omega) \subset X$ and if players' probability density (with respect to μ) over the coming state, ω' is $h(d\omega'|\omega, x)$, given current state ω. Here, for each player d, the integrand, $u_d(\cdot, \cdot, \cdot)$, is Caratheodory, measurable in ω, jointly continuous in (y_d, x), affine in y_d, and quasiconcave in x_d. Thus, under assumptions [A-1], we will be able to show that the game's relevant Nash payoff selection sub-correspondence has the K-limit property and therefore is contractibly-valued - implying that the game's Nash payoff selection correspondence is approximable and therefore has fixed points.

A one-shot game is a collection of parameterized, state-contingent strategic forms games, denoted by

$$\mathcal{G}(\Omega \times \mathcal{L}^{\infty}_{Y}) := \{\mathcal{G}(\omega, v) : (\omega, v) \in \Omega \times \mathcal{L}^{\infty}_{Y}\}, \tag{16}$$

where each (ω, v)-game contained in the one-shot game $\mathcal{G}(\Omega \times \mathcal{L}^{\infty}_{Y})$ is given by

$$\mathcal{G}(\omega, v) := \left(\underbrace{\Phi_d(\omega)}_{\text{constraint set}}, \underbrace{U_d(\omega, v, (\cdot, \cdot))}_{\text{payoff function}} \right)_{d \in D}. \tag{17}$$

3.2 Nash Equilibria

In the strategic form (ω, v)-game, each player $d = 1, 2, \ldots, m$, seeks to choose a feasible action, $x_d \in \Phi_d(\omega)$ so as to maximize $d's$ payoff - i.e., so as to solve the problem

$$\max_{x_d \in \Phi_d(\omega)} U_d(\omega, v_d, (x_d, x_{-d}))$$

$$= \max_{x_d \in \Phi_d(\omega)} \int_\Omega u_d(\omega, v_d^*(\omega'), (x_d, x_{-d}))h(\omega'|\omega, (x_d, x_{-d}))d\mu(\omega'),$$

given current state $\omega \in \Omega$, player $d's$ value function, $v_d \in \mathcal{L}_Y^\infty$, and the feasible actions, x_{-d}, chosen by other players.

A profile of player actions, $x^* = (x_1^*, \ldots, x_m^*) \in \Phi_1(\omega) \times \cdots \times \Phi_m(\omega) := \Phi(\omega)$, is a *Nash equilibrium* for the (ω, v)-game, $\mathcal{G}_{(\omega,v)}$, if for each player $d \in D$

$$U_d(\omega, v_d, (x_d^*, x_{-d}^*)) = \max_{x_d \in \Phi_d(\omega)} U_d(\omega, v_d, (x_d, x_{-d}^*)).$$

3.3 The Three Nash Correspondences

Under assumptions [A-1] for each $(\omega, v) \in \Omega \times \mathcal{L}_Y^\infty$ the (ω, v)-game, $\mathcal{G}(\omega, v)$, has a nonempty, ρ_X-compact set of Nash equilibria, $\mathcal{N}(\omega, v)$, and it is straightforward to show that the *Nash correspondence* (i.e., the first Nash correspondence),

$$\mathcal{N}(\cdot, \cdot) : \Omega \times \mathcal{L}_Y^\infty \longrightarrow P_f(X) \tag{18}$$

belonging to the *collection* of (ω, v)-games (i.e., the one-shot game),

$$\mathcal{G}(\Omega \times \mathcal{L}_Y^\infty) := \{\mathcal{G}(\omega, v) : (\omega, v) \in \Omega \times \mathcal{L}_Y^\infty\}, \tag{19}$$

is *upper Caratheodory*, $(B_\Omega \times B_{w^*}, B_X)$-measurable in (ω, v) and ρ_{w^*}-ρ_X-upper semicontinuous in v.

Given the collection of (ω, v)-games satisfying assumptions [A-1], with upper Caratheodory (uC) Nash correspondence, $(\omega, v) \longrightarrow \mathcal{N}(\omega, v)$, the collection's *Nash payoff correspondence*, $(\omega, v) \longrightarrow \mathcal{P}(\omega, v)$ (i.e., the second Nash correspondence - the composition correspondence), is given by

$$(\omega, v) \longrightarrow \mathcal{P}(\omega, v) := U(\omega, v, \mathcal{N}(\omega, v)), \tag{20}$$

where

$$U(\omega, v, \mathcal{N}(\omega, v)) := \cup_{x \in \mathcal{N}(\omega,v)} \{(U_1(\omega, v_1, x), \ldots, U_m(\omega, v_m, x))\}. \tag{21}$$

The Nash payoff correspondence $\mathcal{P}(\cdot, \cdot)$ induces a *Nash payoff selection correspondence* (i.e., the third Nash correspondence),

$$v \longrightarrow \mathcal{S}^\infty(\mathcal{P}_v) := \mathcal{S}^\infty(\mathcal{P}(\cdot, v)), \tag{22}$$

where $\mathcal{S}^\infty(\mathcal{P}_v)$ is the set of all μ-equivalence classes of measurable functions, $U_{(\cdot)}$, such that $U_\omega \in \mathcal{P}(\omega, v)$ a.e. $[\mu]$. Here, as in our earlier paper, we will show that there exists $v^* \in \mathcal{L}_Y^\infty$ such that $v(\omega) \in \mathcal{P}(\omega, v)$ a.e. $[\mu]$, i.e., such that $v^* \in \mathcal{S}^\infty(\mathcal{P}_{v^*})$.

4 The Error In Our Earlier Paper and Its Correction

In our earlier paper, we considered uC correspondences, $\mathcal{N}(\cdot,\cdot)$, having continuum-valued uC sub-correspondences, $\eta(\cdot,\cdot)$, i.e., continuum-valued uC correspondences such that

$$Gr\eta(\omega,\cdot) \subseteq Gr\mathcal{N}(\omega,\cdot) \text{ a.e. } [\mu]. \tag{23}$$

We will denote by $\mathcal{UC}^{\mathcal{N}}_{C_f(X)}$ the set of all continuum-valued uC sub-correspondences belonging to the uC correspondence, $\mathcal{N}(\cdot,\cdot)$. Here $C_f(X)$ denotes the collection of all nonempty, closed (and hence, compact), connected subsets of X - i.e., the collection of all sub-continua belonging to X). Noting that under assumptions [A-1], if $\eta(\cdot,\cdot) \in \mathcal{UC}^{\mathcal{N}}_{C_f(X)}$, then for each $d = 1, 2, \ldots, m$,

$$(\omega, v) \longrightarrow p_d(\omega, v) := U_d(\omega, v_d, \eta(\omega, v)), \tag{24}$$

player $d's$ Nash payoff sub-correspondence, $p_d(\cdot,\cdot)$, is interval-valued, hence contractibly-valued. We then showed that there exists $v^* \in \mathcal{L}^{\infty}_{Y}$ such that

$$v^*(\omega) \in p_1(\omega, v^*) \times \cdots \times p_1(\omega, v^*) \text{ a.e. } [\mu]. \tag{25}$$

This is all correct - but this is not what we intended to show, nor does (25) allow us to concluded that all convex $DSGs$ have stationary Markov perfect equilibria. Our objective was to show that there exists $v^* \in \mathcal{L}^{\infty}_{Y}$ such that

$$v^*(\omega) \in p(\omega, v^*) := \{(U_1(\omega, v_1^*, x), \ldots, U_m((\omega, v_m^*, x)) : x \in \eta(\omega, v^*)\} \text{ a.e. } [\mu]. \tag{26}$$

But we incorrectly stated that (26) could be deduced from (25) using implicit measurable selection methods (e.g., Theorem 7.1 in Himmelberg, 1975). This is *not* the case. We note that $p(\omega, v^*)$ is a subset of $p_1(\omega, v^*) \times \cdots \times p_1(\omega, v^*)$. Therefore, a $v^* \in \mathcal{L}^{\infty}_{Y}$ satisfying (**??**) does not necessarily satisfy (26). Moreover, in order to conclude that all convex $DSGs$ have stationary Markov perfect equilibria, we must be able to show that there exists $v^* \in \mathcal{L}^{\infty}_{Y}$ satisfying (26). Here we will correct (or more accurately, correctly complete) our earlier paper, and prove the result we intended to prove. We will show that under assumptions [A-1] there exists $v^* \in \mathcal{L}^{\infty}_{Y}$ satisfying (26).

Our focus here, as before, will be on uC Nash payoff sub-correspondences,

$$(\omega, v) \longrightarrow p(\omega, v) = U(\omega, v, \eta(\omega, v)), \tag{27}$$

and their induced Nash payoff selection sub-correspondences,

$$v \longrightarrow \mathcal{S}^{\infty}(p(\cdot, v)) = \mathcal{S}^{\infty}(p_v), \tag{28}$$

where the uC Nash sub-correspondences, $\eta(\cdot,\cdot)$, from which $p(\cdot,\cdot)$ is induced, is *continuum-valued* (ensuring that $p(\cdot,\cdot)$ is continuum-valued), and where $\eta(\cdot,\cdot)$ belongs to a Nash correspondence, $\mathcal{N}(\cdot,\cdot)$, belonging to a one-shot game satisfying assumptions [A-1].[6] Thus, as before our main objective is to show that under assumptions [A-1] the Nash payoff selection sub-correspondence, $\mathcal{S}^{\infty}(p_{(\cdot)})$, induced by the Nash payoff sub-correspondence,

$$(\omega, v) \longrightarrow p(\omega, v) = U(\omega, v, \eta(\omega, v)),$$

for $\eta(\cdot,\cdot) \in \mathcal{UC}^{\mathcal{N}}_{C_f(X)}$, has fixed points.

Summary of Results:

We will show that any such Nash payoff selection sub-correspondence, $\mathcal{S}^{\infty}(p_{(\cdot)})$, has the K-limit property and therefore is a K-correspondence. We say that the Nash payoff selection sub-correspondence, $\mathcal{S}^{\infty}(p_{(\cdot)})$, has the K-limit property if for any sequence $\{(v^n, U^n_{(\cdot)})\}_n \subset Gr\mathcal{S}^{\infty}(p_{(\cdot)})$ K-converging to $(\widehat{v}, \widehat{U}_{(\cdot)}) \in \mathcal{L}^{\infty}_Y \times \mathcal{L}^{\infty}_Y$, where $U^n_{\omega} \in p(\omega, v^n)$ a.e. $[\mu]$ for each n, the K-limit, $\widehat{U}_{(\cdot)}$, is such that $\widehat{U}_{\omega} \in Ls\{U^n_{\omega}\}$ a. e. $[\mu]$. Here, $Ls\{U^n_{\omega}\}$ denotes the set of cluster points of the sequence, $\{U^n_{\omega}\}_n$, in $Y \subset R^m$. Because $p(\omega, \cdot)$ is upper semicontinuous on $\mathcal{L}^{\infty}_Y$ a.e. $[\mu]$, we have for any sequence, $\{(v^n, U^n_{(\cdot)})\}_n \subset Gr\mathcal{S}^{\infty}(p_{(\cdot)})$ K-converging to $(\widehat{v}, \widehat{U}_{(\cdot)})$ that $Ls\{U^n_{\omega}\} \subset p(\omega, \widehat{v})$ a.e. $[\mu]$. Thus, we have for any sequence, $\{(v^n, U^n_{(\cdot)})\}_n \subset Gr\mathcal{S}^{\infty}(p_{(\cdot)})$ K-converging to $(\widehat{v}, \widehat{U}_{(\cdot)})$ that $\mathcal{S}^{\infty}(Ls\{U^n_{(\cdot)}\}) \subset \mathcal{S}^{\infty}(p_{\widehat{v}})$. We show here that for any one-shot game where players have convex, compact metric action sets and player's payoff functions are affinely parameterized Caratheodory functions given by

$$U_d(\omega, v_d, x_d, x_{-d}) := \int_{\Omega} u_d(\omega, v_d(\omega'), x_d, x_{-d}) h(\omega'|\omega, x_d, x_{-d}) d\mu(\omega')$$

satisfying assumptions [A-1] above (including [A-1](11)(iv)), if the dominating probability measure, μ, is nonatomic, then for any sequence,

$$\{(v^n, U^n_{(\cdot)})\}_n \subset Gr\mathcal{S}^{\infty}(p_{(\cdot)})$$

K-converging to $(\widehat{v}, \widehat{U}_{(\cdot)})$,

$$\widehat{U}_{\omega} \in Ls\{U^n_{\omega}\} \subset p(\omega, \widehat{v}) \text{ a.e. } [\mu]. \tag{29}$$

Thus, the K-limit function, $\widehat{U}_{(\cdot)}$, for any such K-converging sequence taken from the graph of the selection sub-correspondence, $Gr\mathcal{S}^{\infty}(p_{(\cdot)})$, is a measurable selection of the Ls-correspondence, $Ls\{U^n_{(\cdot)}\}$ and therefore is contained in $\mathcal{S}^{\infty}(p_{\widehat{v}})$ - implying that the selection sub-correspondence, $v \longrightarrow \mathcal{S}^{\infty}(p_v)$, is upper semicontinuous in the weak star topologies. We then use this fact, together with the decomposability of the collection of μ-equivalence classes of measurable selections, $\mathcal{S}^{\infty}(p_v)$, for each $v \in \mathcal{L}^{\infty}_Y$, and the nonatomicity of the dominating probability measure μ to construct, for each $v \in \mathcal{L}^{\infty}_Y$, a homotopy contracting $\mathcal{S}^{\infty}(p_v)$ onto an arbitrarily chosen selection, $U^1_{(\cdot)}$ of $\mathcal{S}^{\infty}(p_v)$ - thereby showing that, for each $v \in \mathcal{L}^{\infty}_Y$, the collection of μ-equivalence classes, $\mathcal{S}^{\infty}(p_v)$, of selections is contractible. The *contractible* valuedness of $\mathcal{S}^{\infty}(p_v)$

[6] Fu and Page (2022) have shown that all one shot games satisfying assumptions [A-1] possess continuum-valued uC Nash sub-correspondences.

for each $v \in \mathcal{L}_Y^\infty$, in turn implies that the Nash payoff selection sub-correspondence, $\mathcal{S}^\infty(p_{(\cdot)})$, is approximable and has fixed points.

Before we prove our key result on the K-limit property we will review some of the fundamental results we will need concerning the relationship between weak star convergence and K-convergence as well as decomposable sets of measurable selections.

5 w^*-Convergence and K-Convergence in $\mathcal{L}_Y^\infty$

A sequence, $\{v^n\}_n \subset \mathcal{L}_Y^\infty$, converges weak star to $v^* = (v_1^*(\cdot), \ldots, v_m^*(\cdot)) \in \mathcal{L}_Y^\infty$, denoted by $v^n \underset{\rho_{w^*}}{\longrightarrow} v^*$, if and only if

$$\int_\Omega \langle v^n(\omega), l(\omega)\rangle_{R^m}\, d\mu(\omega) \longrightarrow \int_\Omega \langle v^*(\omega), l(\omega)\rangle_{R^m}\, d\mu(\omega) \tag{30}$$

for all $l(\cdot) \in \mathcal{L}_{R^m}^1$.

A sequence, $\{v^n\}_n \subset \mathcal{L}_Y^\infty$, K-convergences (i.e., Komlos convergence - Komlos, 1967) to $v^* \in \mathcal{L}_Y^\infty$, denoted by $v^n \underset{K}{\longrightarrow} v^*$, if and only if every subsequence, $\{v^{n_k}(\cdot)\}_k$, of $\{v^n(\cdot)\}_n$ has an arithmetic mean sequence, $\{\widehat{v}^{n_k}(\cdot)\}_k$, where

$$\widehat{v}^{n_k}(\cdot) := \frac{1}{k}\sum_{q=1}^{k} v^{n_q}(\cdot), \tag{31}$$

such that

$$\widehat{v}^{n_k}(\omega) \underset{R^m}{\longrightarrow} v^*(\omega) \;\; a.e. \;\; [\mu]. \tag{32}$$

The relationship between w^*-convergence and K-convergence is summarized via the following results (see Theorem A 2.1, Page, 2016): For every sequence of value functions, $\{v^n\}_n \subset \mathcal{L}_Y^\infty$, and $\widehat{v} \in \mathcal{L}_Y^\infty$ the following statements are true (i.e., the near equivalence of w^* and K convergence):

$$\left.\begin{array}{c} \text{(i) If the sequence } \{v^n\}_n \ K\text{-converges to } \widehat{v}, \\ \text{then } \{v^n\}_n \ w^*\text{-converges to } \widehat{v}. \\ \\ \text{(ii) The sequence } \{v^n\}_n \ w^*\text{-converges to } v^* \text{ if and only if} \\ \text{every subsequence } \{v^{n_k}\}_k \text{ of } \{v^n\}_n \\ \text{has a further subsequence, } \{v^{n_{k_r}}\}_r,\ K\text{-converging to } v^*. \end{array}\right\} \tag{33}$$

For any sequence of value function profiles, $\{v^n\}_n$, in $\mathcal{L}_Y^\infty$ it is automatic that

$$sup_n \int_\Omega \|v^n(\omega)\|_{R^m}\, d\mu(\omega) < +\infty. \tag{34}$$

Thus, by the classical Komlos Theorem (1967), any such sequence, $\{v^n\}_n$, has a subsequence, $\{v^{n_k}\}_k$ that K-converges to some K-limit, $\widehat{v} \in \mathcal{L}_Y^\infty$. Moreover, by Proposition 1(1) in Page (1991),

$$\widehat{v}(\omega) \in coLs\{v^n(\omega)\} \;\; a.e. \;\; [\mu], \tag{35}$$

and by Proposition 1(2) there exists an integrable R^m-valued function, $v^*(\cdot)$, such that $v^*(\omega) \in Ls\{v^n(\omega)\}$ a.e. $[\mu]$ and

$$\int_\Omega v^*(\omega)d\mu(\omega) = \int_\Omega \widehat{v}(\omega)d\mu(\omega). \tag{36}$$

The fact that the K-limit, $\widehat{v} \in \mathcal{L}^\infty_Y$, is an a.e. selection of the correspondence, $coLs\{v^n(\cdot)\}$, is a key result in our proof that affinely parameterized upper Caratheodory composition correspondences,

$$(\omega, v) \longrightarrow p(\omega, v) := \{(U_1(\omega, v_1, x), \ldots, U_m((\omega, v_m, x)) : x \in \eta(\omega, v)\},$$

induce a selection correspondence, $v \longrightarrow \mathcal{S}^\infty(p_v)$, having the K-limit property.

6 Decomposability in $\mathcal{L}^\infty_Y$

A subset $\mathcal{S}$ of $\mathcal{L}^\infty_Y$ is said to be decomposable if for any two functions $U^0_{(\cdot)}$ and $U^1_{(\cdot)}$ in $\mathcal{S}$ and for any $E \in B_\Omega$, we have

$$U^0_{(\cdot)} I_E(\cdot) + U^1_{(\cdot)} I_{\Omega \backslash E}(\cdot) \in \mathcal{S}.$$

For any uC Nash payoff sub-correspondence, $p(\cdot,\cdot) : \Omega \times \mathcal{L}^\infty_Y \longrightarrow P_f(Y)$, the induced Nash payoff selection correspondence, $\mathcal{S}^\infty(p_{(\cdot)})$, takes decomposable values. Moreover, for each v, $\mathcal{S}^\infty(p_v)$ is $\|\cdot\|_1$-closed (or $\mathcal{L}^1_{R^m}$-closed) in $\mathcal{L}^\infty_{R^m}$. Thus, for any sequence $\{U^n_{(\cdot)}\}_n$ in $\mathcal{S}^\infty(p_v)$ converging in $\mathcal{L}^1_{R^m}$-norm to $U^0_{(\cdot)} \in \mathcal{L}^\infty_{R^m}$, we have $U^0_{(\cdot)} \in \mathcal{S}^\infty(p_v)$. We will denote by $cl_1\mathcal{S}^\infty(p_v)$ the $\mathcal{L}^1_{R^m}$-closure of $\mathcal{S}^\infty(p_v)$ in $\mathcal{L}^\infty_{R^m}$. By Lemma 1 in Pales and Zeidan (1999), we know that, in addition to $\mathcal{S}^\infty(p_v)$ being decomposable, $\mathcal{S}^\infty(p_v)$ is $\mathcal{L}^1_{R^m}$-closed in $\mathcal{L}^\infty_{R^m}$. Thus, we have

$$cl_1\mathcal{S}^\infty(p_v) = \mathcal{S}^\infty(p_v).$$

We also know by Corollary 1 in Pales and Zeidan (1999) that

$$cl_1\mathcal{S}^\infty(p_v)$$

$$= \left\{U_{(\cdot)} \in \mathcal{L}^\infty_{R^m} : \exists\ \{U^n_{(\cdot)}\}_n \ \subset \mathcal{S}^\infty(p_v) \text{ such that } \lim_n \left\|U^n_{(\cdot)} - U_{(\cdot)}\right\|_1 = 0\right\}.$$

Finally, note that $\mathcal{L}^\infty_Y$ is $\mathcal{L}^1_{R^m}$-closed in $\mathcal{L}^\infty_{R^m}$ and decomposable.

7 The K-Limit Property

Let $(\omega, v) \longrightarrow p(\omega, v) := U(\omega, v, \eta(\omega, v))$ be a uC composition sub-correspondence with underlying continuum-valued uC sub-correspondence, $\eta(\cdot,\cdot)$ belonging to uC correspondence, $\mathcal{N}(\cdot,\cdot)$, and let

$$v \longrightarrow \mathcal{S}^\infty(p_v)$$

be the induced selection sub-correspondence. We have the following formal definition of the K-limit property.

Definition 1 *(The K-Limit Property and K-Correspondences):*
We say that the selection sub-correspondence, $\mathcal{S}^{\infty}(p_{(\cdot)})$, has the K-limit property if for any K-converging sequence,

$$\{(v^n, U^n_{(\cdot)})\}_n \subset Gr\mathcal{S}^{\infty}(p_{(\cdot)}),$$

the K limit, $(\widehat{v}, \widehat{U}_{(\cdot)})$, is such that

$$\widehat{U}_{\omega} \in Ls\{U^n_{\omega}\} \subset p(\omega, \widehat{v}) \text{ a.e. } [\mu].$$

Because $\{(v^n, U^n_{(\cdot)})\}_n$ K-converges to K-limit, $(\widehat{v}, \widehat{U}_{(\cdot)})$, $\{v^n\}_n$ ρ_{w^*}-converges to $\widehat{v}$ and $\{U^n_{(\cdot)}\}_n$ ρ_{w^*}-converges to $\widehat{U}_{(\cdot)}$. Moreover, because $p(\omega, \cdot)$ is ρ_{w^*}-ρ_Y-upper semicontinuous, $(v^n, U^n_{\omega}) \in Grp(\omega, \cdot)$ *a.e.* $[\mu]$ for all n, implies that $Ls\{U^n_{\omega}\} \subset p(\omega, \widehat{v})$ *a.e.* $[\mu]$. Thus, if the selection sub-correspondence, $\mathcal{S}^{\infty}(p_{(\cdot)})$, has the K-limit property, then for *any K-converging sequence,*

$$\{(v^n, U^n_{(\cdot)})\}_n \subset \mathcal{L}^{\infty}_Y \times \mathcal{L}^{\infty}_Y,$$

with $(v^n, U^n_{(\cdot)}) \in Gr\mathcal{S}^{\infty}(p_{(\cdot)})$ for each n, the K limit, $(\widehat{v}, \widehat{U}_{(\cdot)}) \in \mathcal{L}^{\infty}_Y \times \mathcal{L}^{\infty}_Y$, is such that

$$\widehat{U}_{(\cdot)} \in \mathcal{S}^{\infty}(Ls\{U^n_{(\cdot)}\}) \subset \mathcal{S}^{\infty}(p_{\widehat{v}}).$$

Let N^{∞} be the exceptional set (i.e., the set of μ-measure zero) such for $\omega \in \Omega \backslash N^{\infty}$, $U^n_{\omega} \in p(\omega, v^n)$ for all n. For each n, we have by the Measurable Implicit Function Theorem (e.g., Himmelberg, 1975, Theorem 7.1) a (B_{Ω}, B_X)-measurable function, $x^n(\cdot) : \Omega \longrightarrow X$, such that for each n and $\omega \in \Omega \backslash N^{\infty}$,

$$U^n_{\omega} = U(\omega, v^n, x^n(\omega)) \in p(\omega, v^n) \text{ with } x^n(\omega) \in \eta(\omega, v^n),$$

and thus,

$$\{(v^n, U(\cdot, v^n, x^n(\cdot))\}_n \subset Gr\mathcal{S}^{\infty}(p_{(\cdot)}) \subset \mathcal{L}^{\infty}_Y \times \mathcal{L}^{\infty}_Y.$$

An alternative statement of the K-limit property is

$\mathcal{S}^{\infty}(p_{(\cdot)})$ has the K-limit property if for any K-converging sequence,

$$\{(v^n, U(\cdot, v^n, x^n(\cdot)))\}_n \subset Gr\mathcal{S}^{\infty}(p_{(\cdot)}) \subset \mathcal{L}^{\infty}_Y \times \mathcal{L}^{\infty}_Y,$$

with K-limit, $(\widehat{v}, \widehat{U}_{(\cdot)}) \in \mathcal{L}^{\infty}_Y \times \mathcal{L}^{\infty}_Y$, the K-limit, $(\widehat{v}, \widehat{U}_{(\cdot)})$, is such that

$$\widehat{U}_{\omega} \in U(\omega, \widehat{v}, Ls\{x^n(\omega)\}) \subset p(\omega, \widehat{v}) \text{ a.e. } [\mu],$$

where

$$U(\omega, \widehat{v}, Ls\{x^n(\omega)\}) := \{U(\omega, \widehat{v}, x) \in Y : x \in Ls\{x^n(\omega)\}\}.$$

We note that $Ls\{U^n_{\omega}\} = U(\omega, \widehat{v}, Ls\{x^n(\omega)\})$.

Now we have our main result on the K-limit property for nonatomic convex $DSGs$

Theorem 1 (*The K-Limit Theorem for nonatomic convex DSGs - the Nice Lemma*)
Let

$$(\omega, v) \longrightarrow p(\omega, v) := U(\omega, v, \eta(\omega, v))$$

$$= \left\{ \left(\int_\Omega u_d(\omega, v_d(\omega'), x(\omega)) h(d\omega'|\omega, x) d\mu(\omega') \right)_{d \in D} : x \in \eta(\omega, v) \right\},$$

be the Nash payoff sub-correspondence induced by any continuum-valued Nash sub-correspondence, $\eta(\cdot, \cdot)$, belonging to a nonatomic convex DSG. Then the induced Nash payoff selection sub-correspondence,

$$v \longrightarrow \mathcal{S}^\infty(p_v) := \mathcal{S}^\infty(p(\cdot, v)) := \mathcal{S}^\infty(u(\cdot, v, \eta(\cdot, v))),$$

has the K-limit property, and therefore, is a K-correspondence.

Proof: Let $\{(v^n, U^n_{(\cdot)})\}_n \subset \mathcal{L}^\infty_Y \times \mathcal{L}^\infty_Y$ be any K-converging sequence with K-limit, $(\widehat{v}, \widehat{U}_{(\cdot)}) \in \mathcal{L}^\infty_Y \times \mathcal{L}^\infty_Y$, where for each n and ω a.e. $[\mu]$,

$$\left.\begin{array}{c} U^n_\omega = U(\omega, v^n, x^n(\omega)) \in p(\omega, v^n) \\ \text{and} \\ x^n(\omega) \in \eta(\omega, v^n), \end{array}\right\} \tag{37}$$

with

$$\begin{aligned} U^n_\omega &= U(\omega, v^n, x^n(\omega)) \\ &= \left(\int_\Omega u_d(\omega, v^n_d(\omega'), x^n(\omega)) h(\omega'|\omega, x^n(\omega)) d\mu(\omega') \right)_{d \in D}. \end{aligned} \tag{38}$$

Let

$$\widehat{v}^n(\cdot) := \frac{1}{n} \sum_{k=1}^n \widehat{v}^k(\cdot) \text{ and } \widehat{U}^n_{(\cdot)} := \frac{1}{n} \sum_{k=1}^n \widehat{U}^k_{(\cdot)} \tag{39}$$

denote the arithmetic mean functions induced by the sequences, $\{v^n\}_n$ and $\{U^n_{(\cdot)}\}_n$. Because $\{v^n\}_n \subset \mathcal{L}^\infty_Y$ K-converges to $\widehat{v} \in \mathcal{L}^\infty_Y$, any sequence of arithmetic mean functions, $\{\widehat{v}^{n_k}\}_k$, belonging to any subsequence, $\{v^{n_k}\}_k$, of $\{v^n\}_n$ converges pointwise a.e. to $\widehat{v}$, implying that the sequence itself, $\{v^n\}_n$, ρ_{w^*}-converges to $\widehat{v}$. Next let $x^*(\cdot)$ be an everywhere measurable selection of the Ls correspondence, $\omega \longrightarrow Ls\{x^n(\omega)\}$, i.e., $x^*(\omega) \in Ls\{x^n(\omega)\}$ for all $\omega \in \Omega$. By the Kuratowski-Ryll-Nardzewski Theorem (1965) we know such a measurable selection exists. Let $\widehat{N}$ be the exceptional set for the K-converging sequence, $\{(v^n, U^n_{(\cdot)})\}_n$. Thus, for $\omega \in \Omega \backslash \widehat{N}$,

$$(\widehat{v}^n(\omega), \widehat{U}^n_\omega) \longrightarrow (\widehat{v}(\omega), \widehat{U}_\omega) \in Y \times Y,$$

keeping in mind that for a subsequence of arithmetic mean function, $\{(\widehat{v}^{n_k}(\cdot), \widehat{U}^{n_k}_{(\cdot)})\}_k$, induced by a subsequence, $\{(v^{n_k}(\cdot), U^{n_k}_{(\cdot)})\}_k$ of $\{(v^n(\cdot), U^n_{(\cdot)})\}_n$, the exceptional set may differ from $\widehat{N}$.

Under assumptions [A-1], we have for any $\omega \in \Omega\backslash\widehat{N}$, some subsequence, $\{x^{n_k}(\omega)\}_k$, ρ_X-converges to $x^*(\omega)$, and because we also have $v_d^{n_k} \underset{\rho_{w_d^*}}{\longrightarrow} \widehat{v}_d$ and

$$h(\omega'|\omega, x^{n_k}(\omega)) \longrightarrow h(\omega'|\omega, x^*(\omega)) \text{ with } \|h(\cdot|\omega, x^{n_k}(\omega)) - h(\cdot|\omega, x^*(\omega))\|_1 \longrightarrow 0,$$

we know that for $d = 1, 2, \ldots, m,$

$$\left.\begin{array}{l} \int_\Omega u_d(\omega, v_d^{n_k}(\omega'), x^{n_k}(\omega))h(\omega'|\omega, x^{n_k}(\omega))d\mu(\omega') \\ \longrightarrow \int_\Omega u_d(\omega, \widehat{v}_d(\omega'), x^*(\omega))h(\omega'|\omega, x^*(\omega))d\mu(\omega'). \end{array}\right\} \tag{40}$$

Next let

$$G^n(\omega, \omega') := (u_d(\omega, v_d^n(\omega'), x^n(\omega))h(\omega'|\omega, x^n(\omega)))_d. \tag{41}$$

Because $|u_d(\omega, v_d^n(\omega'), x^n(\omega))| \leq M$ and because $h(\cdot|\omega, x^n(\omega))$ is a probability density, we have for all d, n and (ω, ω') that the sequence of functions, $\{G^n(\cdot,\cdot)\}_n \subset \mathcal{L}^1_{R^m}(\Omega \times \Omega)$, is norm bounded. We have for all n,

$$U^n_\omega := \int_\Omega G^n(\omega, \omega')d\mu(\omega') := \left(\int_\Omega u_d(\omega, v_d^n(\omega'), x^n(\omega))h(\omega'|\omega, x^n(\omega))d\mu(\omega')\right)_d. \tag{42}$$

Thus,

$$\left.\begin{array}{l} Ls\{U^n_\omega\} := \left\{\left(\int_\Omega u_d(\omega, \widehat{v}_d(\omega'), x^*(\omega))h(\omega'|\omega, x^*(\omega))d\mu(\omega')\right)_d : x^*(\omega) \in Ls\{x^n(\omega)\}\right\} \\ \qquad := Ls\{\int_\Omega G^n(\omega, \omega')d\mu(\omega')\}. \end{array}\right\} \tag{43}$$

Also applying Komlos (1967), we can assume without loss of generality that the norm bounded sequence of R^m-valued integrable functions, $\{G^n(\cdot,\cdot)\}_n$ K-converges to K-limit $\widehat{G}(\cdot,\cdot) \in \mathcal{L}^1_{R^m}(\Omega \times \Omega)$.[7] Therefore, for ω off of some exceptional set, $\widehat{N}_G$, we

[7] Here, each $G^n(\cdot,\cdot)$ is an integrable, R^m-valued function defined on the product probability space,

$$(\Omega \times \Omega, B_\Omega \times B_\Omega, \mu \otimes \mu).$$

Let $\lambda := \mu \otimes \mu$ and let

$$\widehat{G}^n(\cdot,\cdot) := \frac{1}{n}\sum_{k=1}^{n} G^n(\cdot,\cdot).$$

By K-convergence, we have that $\widehat{G}^n(\omega, \omega') \longrightarrow_{R^m} \widehat{G}(\omega, \omega')$ for (ω, ω') off an exceptional set $\widehat{E} \in B_\Omega \times B_\Omega$ with $\lambda(\widehat{E}) = 0$. We have for the exceptional set $\widehat{E}$ that

$$\lambda(\widehat{N}) = \int_\Omega \mu(\widehat{E}(\omega))d\mu(\omega) = 0,$$

where

$$\widehat{E}(\omega) := \left\{\omega' \in \Omega : (\omega, \omega') \in \widehat{E}\right\},$$

implying that for some $\widehat{N}_G$ with $\mu(\widehat{N}_G) = 0$, $\mu(\widehat{E}(\omega)) = 0$ for all $\omega \in \Omega\backslash\widehat{N}_G$ Thus for each $\omega \in \Omega\backslash\widehat{N}_G$

$$\widehat{G}^n(\omega, \omega') \longrightarrow \widehat{G}(\omega, \omega') \text{ for } \omega' \text{ a.e. } [\mu],$$

(see Ash, 1972).

have that

$$\widehat{G}^n(\omega,\omega') := \frac{1}{n}\sum_{k=1}^{n} G^k(\omega,\omega') \underset{R^m}{\longrightarrow} \widehat{G}(\omega,\omega') \text{ a.e. } [\mu] \text{ in } \omega', \tag{44}$$

and therefore for $\omega \in \Omega \backslash \left(\widehat{N} \cup \widehat{N}_G\right)$,

$$\left.\begin{array}{c} \widehat{U}^n_\omega := \frac{1}{n}\sum_{k=1}^{n}\int_\Omega G^k(\omega,\omega')d\mu(\omega') \\ = \int_\Omega \frac{1}{n}\sum_{k=1}^{n} G^k(\omega,\omega')d\mu(\omega') \underset{R^m}{\longrightarrow} \int_\Omega \widehat{G}(\omega,\omega')d\mu(\omega') = \widehat{U}_\omega. \end{array}\right\} \tag{45}$$

Next, consider the auxiliary function

$$F^n(\omega,\omega') := \left(u_d(\omega,\widehat{v}_d(\omega'),x^n(\omega))h(\omega'|\omega,x^n(\omega))\right)_d \tag{46}$$

where for each d the value function, $v^n_d(\cdot)$, in the definition of the function $G^n(\cdot,\cdot)$ (see expression (41)) is replaced by the K-limit function, $\widehat{v}_d(\cdot)$, in the definition of the function $F^n(\cdot,\cdot)$ in expression (46). Thus, by Remarks 1 and 2 we have for any measurable selection, $x^*(\cdot)$, of the Ls correspondence, $Ls\{x^n(\cdot)\}$, for any $\omega \in \Omega \backslash \left(\widehat{N} \cup \widehat{N}_G\right)$, and for any subsequence, $\{x^{n_k}(\omega)\}_k$, such that $x^{n_k}(\omega) \underset{\rho_X}{\longrightarrow} x^*(\omega)$, that

$$u_d(\omega,\widehat{v}_d(\cdot),x^{n_k}(\omega)) \underset{\rho_{w^*_d}}{\longrightarrow} u_d(\omega,\widehat{v}_d(\cdot),x^*(\omega)) \text{ for each } d, \tag{47}$$

and

$$h(\omega'|\omega,x^{n_k}(\omega)) \longrightarrow h(\omega'|\omega,x^*(\omega)) \text{ with } \|h(\cdot|\omega,x^{n_k}(\omega)) - h(\cdot|\omega,x^*(\omega))\|_1 \longrightarrow 0,$$

we because $\{h(\cdot|\omega,x^n(\omega))\}_n \subset \mathcal{L}^1_R$ is uniformly integrable, we have for each d and each $\omega \in \Omega \backslash \left(\widehat{N} \cup \widehat{N}_G\right)$,

$$\left.\begin{array}{c} \int_\Omega F^{n_k}(\omega,\omega')d\mu(\omega') = \int_\Omega u_d(\omega,\widehat{v}_d(\omega'),x^{n_k}(\omega))h(\omega'|\omega,x^{n_k}(\omega))d\mu(\omega') \\ \longrightarrow \int_\Omega u_d(\omega,\widehat{v}_d(\omega'),x^*(\omega))h(\omega'|\omega,x^*(\omega))d\mu(\omega') \in Ls\{\int_\Omega F^n(\omega,\omega')d\mu(\omega')\}. \end{array}\right\} \tag{48}$$

We also have for any $\omega \in \Omega \backslash \left(\widehat{N} \cup \widehat{N}_G\right)$,

$$\left.\begin{array}{c} u_d(\omega,\widehat{v}_d(\omega'),x^{n_k}(\omega))h(\omega'|\omega,x^{n_k}(\omega)) \\ \underset{R}{\longrightarrow} u_d(\omega,\widehat{v}_d(\omega'),x^*(\omega))h(\omega'|\omega,x^*(\omega)) \in Ls\{F^n(\omega,\omega')\} \text{ a.e. } [\mu] \text{ in } \omega', \\ \text{so that} \\ \int_\Omega u_d(\omega,\widehat{v}_d(\omega'),x^*(\omega))h(\omega'|\omega,x^*(\omega))d\mu(\omega') \in \int_\Omega Ls\{F^n(\omega,\omega')\}d\mu(\omega') \end{array}\right\} \tag{49}$$

Thus, we have for the sequence of auxiliary function values, $\{F^n(\omega,\omega')\}_n$, and the sequence, $\{G^n(\omega,\omega')\}_n$ that for any $\omega \in \Omega \backslash \left(\widehat{N} \cup \widehat{N}_G\right)$,

$$\left.\begin{array}{c} Ls\{\int_\Omega F^n(\omega,\omega')d\mu(\omega')\} = \int_\Omega Ls\{F^n(\omega,\omega')\}d\mu(\omega'), \\ \text{and} \\ Ls\{\int_\Omega F^n(\omega,\omega')d\mu(\omega')\} = Ls\{\int_\Omega G^n(\omega,\omega')d\mu(\omega')\}. \end{array}\right\} \tag{50}$$

Finally, by Page (1991) Proposition 1(1), we have that a.e. $[\mu]$ in ω

$$\widehat{U}_\omega \in coLs\{U^n_\omega\} = coLs\{\int_\Omega G^n(\omega, \omega')d\mu(\omega')\}, \tag{51}$$

and therefore by (50) we have for ω off the exceptional set that

$$\left.\begin{aligned} \widehat{U}_\omega \in coLs\{U^n_\omega\} &= coLs\{\textstyle\int_\Omega G^n(\omega, \omega')d\mu(\omega')\} \\ &= coLs\{\textstyle\int_\Omega F^n(\omega, \omega')d\mu(\omega')\} \\ &= co\textstyle\int_\Omega Ls\{F^n(\omega, \omega')\}d\mu(\omega'). \end{aligned}\right\} \tag{52}$$

By the properties of Aumann integrals over nonatomic probability spaces (see Hildenbrand, 1974), we have that

$$co\int_\Omega Ls\{F^n(\omega, \omega')\}d\mu(\omega') = \int_\Omega Ls\{F^n(\omega, \omega')\}d\mu(\omega'), \tag{53}$$

and again by (50), we have that

$$\int_\Omega Ls\{F^n(\omega, \omega')\}d\mu(\omega') = Ls\{\int_\Omega F^n(\omega, \omega')d\mu(\omega')\}, \tag{54}$$

implying via (50) that

$$Ls\{\int_\Omega F^n(\omega, \omega')d\mu(\omega')\} = Ls\{\int_\Omega G^n(\omega, \omega')d\mu(\omega')\} = Ls\{U^n_\omega\}. \tag{55}$$

Thus, by Proposition 1 in Page (1991) and (50)-(55), we have that

$$\widehat{U}_\omega \in coLs\{U^n_\omega\} = coLs\{\textstyle\int_\Omega G^n(\omega, \omega')d\mu(\omega')\}$$

$$= Ls\{\textstyle\int_\Omega G^n(\omega, \omega')d\mu(\omega')\} = Ls\{U^n_\omega\}.$$

We can conclude, therefore, that in a nonatomic, convex DSG, if we are given any K-converging sequence $\{(v^n, U^n_{(\cdot)})\}_n \subset \mathcal{S}^\infty(p(\cdot, v^n)) := \mathcal{S}^\infty(u(\cdot, v^n, \eta(\cdot, v^n)))$, with K-limit $(\widehat{v}, \widehat{U}_{(\cdot)})$, where for each n

$$U^n_\omega = u(\omega, v^n, x^n(\omega)) \in p(\omega, v^n) \text{ and } x^n(\omega) \in \eta(\omega, v^n) \text{ a.e. } [\mu],$$

then there exists for each ω, off some exceptional set of measure zero, a $U^*_\omega \in Ls\{U^n_\omega\}$ such that $U^*_\omega = \widehat{U}_\omega$. Thus, the selection sub-correspondence, $\mathcal{S}^\infty(p_{(\cdot)})$, has the K-limit property, and therefore, is a K-correspondence. **Q.E.D.**

By the Theorem 1 above, the K-limit $\widehat{U}_{(\cdot)}$ of a K-converging sequence of a.e. selections, $\{U^n_{(\cdot)}\}_n$, of $p(\cdot, v^n)$ for each n is an a.e. measurable selection of the Ls correspondence, $Ls\{U^n_{(\cdot)}\}$, induced by the sequence. As we show below, the K-limit property, a property inherit to all nonatomic convex $DSGs$ with player payoff functions given in expression (38), is sufficient to guarantee that all nonatomic convex $DSGs$ have Nash payoff selection correspondences having fixed points.

8 A Fixed Point Theorem for Nonatomic Convex $DSGs$

Let $(\omega, v) \longrightarrow p(\omega, v) := u(\omega, v, \eta(\omega, v))$ be a Nash payoff sub-correspondence induced by a continuum-valued Nash sub-correspondence belonging to a nonatomic convex DSG. We will show that because the uC Nash payoff selection sub-correspondence, $\mathcal{S}^{\infty}(p_{(\cdot)})$, induced by any continuum-valued Nash sub-correspondence belonging to a nonatomic convex DSG has the K-limit property, *$\mathcal{S}^{\infty}(p_{(\cdot)})$ is a ρ_{w^*}-ρ_{w^*}-USCO taking contractible values* - implying that $\mathcal{S}^{\infty}(p_{(\cdot)})$ is ρ_{w^*}-ρ_{w^*}-approximable and therefore that $\mathcal{S}^{\infty}(p_{(\cdot)})$ has fixed points.[8]

8.1 The Contractibility Result

Theorem 2 *($\mathcal{S}^{\infty}(p_{(\cdot)})$ is a ρ_{w^*}-ρ_{w^*}-USCO taking contractible values in $\mathcal{L}^{\infty}_{Y}$)*
Let

$$(\omega, v) \longrightarrow p(\omega, v) := u(\omega, v, \eta(\omega, v))$$

be the Nash payoff sub-correspondence induced by any continuum-valued Nash sub-correspondence belonging to a nonatomic convex DSG. Then $\mathcal{S}^{\infty}(p_{(\cdot)})$ is a ρ_{w^*}-ρ_{w^*}-USCO and for each $v \in \mathcal{L}^{\infty}_{Y}$, $\mathcal{S}^{\infty}(p_v)$ is contractible.

Proof: First, by Theorem 1 above, $\mathcal{S}^{\infty}(p_{(\cdot)})$ is a K-correspondence, under assumptions $[OSG](1)$-(11) it follows from Komlos (1967) and Page (1991) that for each $v \in \mathcal{L}^{\infty}_{Y}$, $\mathcal{S}^{\infty}(p_v)$ is ρ_{w^*}-compact. Therefore, to show that $\mathcal{S}^{\infty}(p_{(\cdot)})$ is a ρ_{w^*}-ρ_{w^*}-$USCO$, it suffices to show that $Gr\mathcal{S}^{\infty}(p_{(\cdot)})$ is $\rho_{w^* \times w^*}$-closed in $\mathcal{L}^{\infty}_{Y} \times \mathcal{L}^{\infty}_{Y}$. Let $\{(v^n, U^n_{(\cdot)})\}_n$ be any sequence in $Gr\mathcal{S}^{\infty}(p_{(\cdot)})$ such that $v^n \underset{\rho_{w^*}}{\longrightarrow} v^*$ and $U^n_{(\cdot)} \underset{\rho_{w^*}}{\longrightarrow} U^*_{(\cdot)}$. We have then a subsequence, $\{(v^{n_k}, U^{n_k}_{(\cdot)})\}_k$, such that

$$v^{n_k} \underset{K}{\longrightarrow} \widehat{v} \text{ and } U^{n_k}_{(\cdot)} \underset{K}{\longrightarrow} \widehat{U}_{(\cdot)}, \text{ with } \widehat{v}(\omega) = v^*(\omega) \text{ and } \widehat{U}_{\omega} = U^*_{\omega} \text{ a.e. } [\mu].$$

Because $\mathcal{S}^{\infty}(p_{(\cdot)})$ is a K-correspondence, $(\widehat{v}, \widehat{U}_{(\cdot)}) \in Gr\mathcal{S}^{\infty}(p_{(\cdot)})$. Thus, we have $(v^*, U^*_{(\cdot)}) \in Gr\mathcal{S}^{\infty}(p_{(\cdot)})$.

Next, for $\mathcal{S}^{\infty}(p_{(\cdot)})$ a ρ_{w^*}-ρ_{w^*}-USCO, we will show that because the dominating probability measure, μ, is nonatomic, for each v, $\mathcal{S}^{\infty}(\mathcal{P}_v)$ is contractible.

As shown by Fryszkowski (1983), if μ is nonatomic Lyapunov's Theorem (1940) on the range of a vector measure guarantees the existence of a family of measurable sets, $\{E_t\}_{t \in [0,1]}$, such that

$$\left.\begin{array}{c} t' \leq t \Rightarrow E_{t'} \subseteq E_t,\ E_0 = \varnothing \text{ and } E_1 = \Omega, \text{ and} \\ \mu(E_t) = t\mu(\Omega) = t. \end{array}\right\} \tag{56}$$

Using the properties of this system of measurable sets and the decomposability of $\mathcal{S}^{\infty}(p_v)$ for each $v \in \mathcal{L}^{\infty}_{Y}$, we will show that for each v the function $h_v(\cdot, \cdot)$ given by

$$h_v(U_{(\cdot)}, t) := U^1_{(\cdot)} I_{E_t}(\cdot) + U_{(\cdot)} I_{\Omega \backslash E_t}(\cdot) \in \mathcal{S}^{\infty}(p_v) \text{ for all } (U_{(\cdot)}, t) \in \mathcal{S}^{\infty}(p_v) \times [0,1] \tag{57}$$

[8] An $USCO$ is an upper semicontinuous correspondence taking nonempty, compact values (e.g., Hola and Holy, 2015).

is a homotopy (and in particular, a contraction of $\mathcal{S}^\infty(p_v)$ to $U^1_{(\cdot)}$). Here $v \in \mathcal{L}^\infty_Y$ is fixed, $I_E(\cdot)$ is the indicator function of set E, and $U^1_{(\cdot)}$ is any fixed selection in $\mathcal{S}^\infty(p_v)$.

It suffices to show that $h_v(\cdot,\cdot)$ is $\rho_{w^*\times|\cdot|}$-ρ_{w^*}-continuous. Let $\{(U^n_{(\cdot)}, t^n)\}_n$ be a sequence such that

$$U^n_{(\cdot)} \underset{\rho_{w^*}}{\longrightarrow} U^*_{(\cdot)} \text{ and } t^n \underset{R}{\longrightarrow} t^*.$$

We must show that

$$h_v(U^n_{(\cdot)}, t^n) \underset{\rho_{w^*}}{\longrightarrow} h_v(U^*_{(\cdot)}, t^*) \in \mathcal{S}^\infty(p_v). \tag{58}$$

Rewriting and substituting, we must show that for all $l \in \mathcal{L}^1_{R^m}$,

$$\left.\begin{aligned} H = \underbrace{\int_\Omega \left\langle \left(U^1_\omega I_{E_{t^n}}(\omega) - U^1_\omega I_{E_{t^*}}(\omega)\right), l(\omega)\right\rangle d\mu(\omega)}_{(a)} \\ + \underbrace{\int_\Omega \left\langle \left(U^n_\omega I_{\Omega\setminus E_{t^n}}(\omega) - U^*_\omega I_{\Omega\setminus E_{t^*}}(\omega)\right), l(\omega)\right\rangle d\mu(\omega)}_{(b)} \longrightarrow 0. \end{aligned}\right\} \tag{59}$$

We have for (59)(b) that

$$\left.\begin{aligned} \underbrace{\int_\Omega \left\langle \left(U^n_\omega I_{\Omega\setminus E_{t^n}}(\omega) - U^*_\omega I_{\Omega\setminus E_{t^*}}(\omega)\right), l(\omega)\right\rangle d\mu(\omega)}_{(b)} \\ = \textstyle\int_\Omega \left\langle U^n_\omega, l(\omega) I_{\Omega\setminus E_{t^n}}(\omega)\right\rangle d\mu(\omega) - \int_\Omega \left\langle U^*_\omega, l(\omega) I_{\Omega\setminus E_{t^*}}(\omega)\right\rangle d\mu(\omega). \end{aligned}\right\} \tag{60}$$

Because the sequence $\{l(\cdot)I_{\Omega\setminus E_{t^n}}(\cdot)\}_n \subset \mathcal{L}^1_{R^m}$ $\|\cdot\|_1$-converges to $l(\cdot)I_{\Omega\setminus E_{t^*}}(\cdot) \in \mathcal{L}^1_{R^m}$, the fact that $U^n_{(\cdot)} \underset{\rho_{w^*}}{\longrightarrow} U^*_{(\cdot)}$ implies that

$$\textstyle\int_\Omega \left\langle U^n_\omega, l(\omega) I_{\Omega\setminus E_{t^n}}(\omega)\right\rangle d\mu(\omega) \longrightarrow \int_\Omega \left\langle U^*_\omega, l(\omega) I_{\Omega\setminus E_{t^*}}(\omega)\right\rangle d\mu(\omega). \tag{61}$$

Thus, in expression (59), $(b) \longrightarrow 0$. Similarly, for (59)(a) we have that

$$\left.\begin{aligned} \underbrace{\int_\Omega \left\langle \left(U^1_\omega I_{E_{t^n}}(\omega) - U^1_\omega I_{E_{t^*}}(\omega)\right), l(\omega)\right\rangle d\mu(\omega)}_{(a)} \\ = \textstyle\int_\Omega \left\langle U^1_\omega, l(\omega) I_{E_{t^n}}(\omega)\right\rangle d\mu(\omega) - \int_\Omega \left\langle U^1_\omega, l(\omega) I_{E_{t^*}}(\omega)\right\rangle d\mu(\omega), \end{aligned}\right\} \tag{62}$$

and because the sequence $\{l(\cdot)I_{E_{t^n}}(\cdot)\}_n \subset \mathcal{L}^1_{R^m}$ $\|\cdot\|_1$-converges to $l(\cdot)I_{E_{t^*}}(\cdot) \in \mathcal{L}^1_{R^m}$, we have that

$$\int_\Omega \left\langle U^1_\omega, l(\omega) I_{E_{t^n}}(\omega)\right\rangle d\mu(\omega) \longrightarrow \int_\Omega \left\langle U^1_\omega, l(\omega) I_{E_{t^*}}(\omega)\right\rangle d\mu(\omega). \tag{63}$$

We have then, in expression (59), $(a) \longrightarrow 0$.

Together, (61) and (63) imply that (58) holds. Thus, given the properties of the Lyapunov system (56), the function given in (57) is, for each $v \in \mathcal{L}_Y^\infty$, $\rho_{w^* \times |\cdot|}$-$\rho_{w^*}$-continuous, and therefore, specifies a homotopy for the set of measurable selections, $\mathcal{S}^\infty(p_v)$ - and thus for each v, $\mathcal{S}^\infty(p_v)$ is contractible. **Q.E.D.**

Our proof that $\mathcal{S}^\infty(p_v)$ is contractible for each v is a modified version of the proof given by Mariconda (1992) showing that if the underlying probability space is nonatomic then any decomposable subset of E-valued, Bochner integrable functions in $\mathcal{L}_E^1$ is contractible (where E is a Banach space). In Mariconda's result, the space of functions is equipped with the norm in $\mathcal{L}_E^1$, while here our space of functions (with each function taking values in $Y \subset R^m$) is equipped with the metric, ρ_{w^*}, compatible with the w^* topology.

8.2 The Approximability and Fixed Point Results

The importance of the K-limit property in a nonatomic probability space derives from the fact that it guarantees that $\mathcal{S}^\infty(p_{(\cdot)})$ is *a* ρ_{w^*}-ρ_{w^*}-USCO taking contractible values. This in turn guarantees approximability and the existence of fixed points, as our next results show.

Theorem 3 *($\mathcal{S}^\infty(p_{(\cdot)})$ is ρ_{w^*}-ρ_{w^*}-approximable)*
Let

$$(\omega, v) \longrightarrow p(\omega, v) := u(\omega, v, \eta(\omega, v))$$

be the Nash payoff sub-correspondence induced by any continuum-valued Nash sub-correspondence belonging to a nonatomic convex DSG. Then $\mathcal{S}^\infty(p_{(\cdot)})$ is a ρ_{w^*}-ρ_{w^*}-approximable.

Proof: Because $\mathcal{S}^\infty(p_{(\cdot)})$ is a contractibly-valued ρ_{w^*}-ρ_{w^*}-USCO, by Corollary 5.6 in Gorniewicz, Granas, and Kryszewski (1991), because $\mathcal{S}^\infty(p_{(\cdot)})$ is defined on the ANR (absolute neighborhood retract) space of value functions $\mathcal{L}_Y^\infty$ and takes nonempty, compact, and contractible values in $\mathcal{L}_Y^\infty$ (and hence ∞-proximally connected values - see Theorem 5.3 in Gorniewicz, Granas, and Kryszewski, 1991), $\mathcal{S}^\infty(p_{(\cdot)})$ is a J mapping. Therefore, by Theorem 5.12 in Gorniewicz, Granas, and Kryszewski (1991), $\mathcal{S}^\infty(p_{(\cdot)})$ is ρ_{w^*}-ρ_{w^*}-approximable. **Q.E.D.**

We can now state our main fixed point result.

Theorem 4 *($\mathcal{S}^\infty(p_{(\cdot)})$ has fixed points)*
Let

$$(\omega, v) \longrightarrow p(\omega, v) := u(\omega, v, \eta(\omega, v))$$

be the Nash payoff sub-correspondence induced by any continuum-valued Nash sub-correspondence belonging to a nonatomic convex DSG. Then $\mathcal{S}^\infty(p_{(\cdot)})$ has a fixed point (i.e., there exists $v^* \in \mathcal{L}_Y^\infty$ such that $v^* \in \mathcal{S}^\infty(p_{v^*})$).

Proof: By Theorem 3 above $\mathcal{S}^\infty(p_{(\cdot)})$ is ρ_{w^*}-ρ_{w^*}-approximable. Therefore, we have for each n, a ρ_{w^*}-ρ_{w^*}-continuous function, $g^n(\cdot) : \mathcal{L}_Y^\infty \longrightarrow \mathcal{L}_Y^\infty$, such that for each $(v^n, U^n_{(\cdot)}) \in Grg^n \subset \mathcal{L}^\infty_Y \times \mathcal{L}^\infty_Y$ (i.e., for each $(v^n, U^n_{(\cdot)}) \in \mathcal{L}^\infty_Y \times \mathcal{L}^\infty_Y$, with $U^n_{(\cdot)} = g^n(v^n) \in \mathcal{L}^\infty_Y$) there exists $(\overline{v}^n, \overline{U}^n_{(\cdot)}) \in Gr\mathcal{S}^\infty(p_{(\cdot)})$ such that

$$\rho_{w^*}(v^n, \overline{v}^n) + \rho_{w^*}(U^n_{(\cdot)}, \overline{U}^n_{(\cdot)}) < \tfrac{1}{n}. \tag{64}$$

Equivalently, for any positive integer, n, $Grg^n \subset B_{w^* \times w^*}(\frac{1}{n}, Gr\mathcal{S}^\infty(p_{(\cdot)}))$. Thus, the graph of the continuous function $g^n : \mathcal{L}^\infty_Y \longrightarrow \mathcal{L}^\infty_Y$ is contained in the $\rho_{w^* \times w^*}$-open ball of radius $\frac{1}{n}$ about the graph of $\mathcal{S}^\infty(p_{(\cdot)})$.

Because each of the functions, g^n, is ρ_{w^*}-ρ_{w^*}-continuous and defined on the ρ_{w^*}-compact and convex subset, $\mathcal{L}^\infty_Y$, in $\mathcal{L}^\infty_{R^m}$, taking values in $\mathcal{L}^\infty_Y$, it follows from the fixed point theorem of Schauder (see Aliprantis and Border, 2006), that each g^n has a fixed point, $v^n \in \mathcal{L}^\infty_Y$ (i.e., for each n there exists some $v^n \in \mathcal{L}^\infty_Y$ such that $v^n = g^n(v^n)$). Let $\{v^n\}_n$ be a fixed point sequence corresponding to the sequence of ρ_{w^*}-ρ_{w^*}-continuous approximating functions, $\{g^n(\cdot)\}_n$. Expression (64) can now be rewritten as follows: for each v^n in the fixed point sequence, there is a corresponding pair, $(\overline{v}^n, \overline{U}^n_{(\cdot)}) \in Gr\mathcal{S}^\infty(p_{(\cdot)})$, such that

$$\rho_{w^*}(v^n, \overline{v}^n) + \rho_{w^*}(g^n(v^n), \overline{U}^n_{(\cdot)}) < \tfrac{1}{n},$$

and therefore such that

$$\underbrace{\rho_{w^*}(v^n, \overline{v}^n)}_{A} + \underbrace{\rho_{w^*}(v^n, \overline{U}^n_{(\cdot)})}_{B} < \tfrac{1}{n}. \tag{65}$$

By the ρ_{w^*}-compactness of $\mathcal{L}^\infty_Y$, we can assume WLOG that the fixed point sequence, $\{v^n\}_n \subset \mathcal{L}^\infty_Y$, ρ_{w^*}-converges to a limit $v^* \in \mathcal{L}^\infty_Y$. Thus, by part A of (65), as $n \longrightarrow \infty$ we have

$$v^n \underset{\rho_{w^*}}{\longrightarrow} v^* \text{ and } \overline{v}^n \underset{\rho_{w^*}}{\longrightarrow} v^*,$$

and therefore by part B of (65), as $n \longrightarrow \infty$ we have

$$\overline{U}^n_{(\cdot)} \underset{\rho_{w^*}}{\longrightarrow} v^*.$$

Because $\mathcal{S}^\infty(p_{(\cdot)})$ is $\rho_{w^* \times w^*}$-closed in $\mathcal{L}^\infty_Y \times \mathcal{L}^\infty_Y$,

$$\{(\overline{v}^n, \overline{U}^n_{(\cdot)})\}_n \subset Gr\mathcal{S}^\infty(p_{(\cdot)}),$$

and $\overline{v}^n \underset{\rho_{w^*}}{\longrightarrow} v^*$ and $\overline{U}^n_{(\cdot)} \underset{\rho_{w^*}}{\longrightarrow} v^*$ imply that $(v^*, v^*) \in \mathcal{S}^\infty(p_{(\cdot)})$. Therefore, $v^* \in \mathcal{S}^\infty(p_{v^*})$. **Q.E.D.**

References

[1] Aliprantis, C. D. and Border, K. C. (2006) *Infinite Dimensional Analysis: A Hitchhiker's Guide*, 3rd Edition, Springer-Verlag, Berlin-Heidelberg.

[2] Ash R. (1972) *Probability and Real Analysis*, John Wiley & Sons, New York.

[3] Billingsley, P. (1986) *Probability and Measure*, 2nd Edition, John Wiley & Sons, New York.

[4] Blackwell, D. (1965) "Discounted Dynamic Programming," *Annals of Mathematical Statistics* 36, 226-235.

[5] Fort, M. K. Jr., (1950) "Essential and Nonessential Fixed Points," *American Journal of Mathematics* 72, 315-322.

[6] Fryszkowski, A. (1983) "Continuous Selections from a Class of Non-convex Multivalued Maps," *Studia Math.* 76, 163-174.

[7] Fu, J. and Page, F. (2022a) "Parameterized State-Contingent Games, 3M minimal Nash Correspondences, and Connectedness," Discussion Paper 116, Systemic Risk Centre, London School of Economics.

[8] Fu, J. and Page, F. (2022b) "A Fixed Point Theorem for Measurable Selection Valued Correspondences Induced by Upper Caratheodory Correspondences," Discussion Paper 115, Systemic Risk Centre, London School of Economics.

[9] Fu, J. and Page, F. (2023) "A Fixed Point Theorem for Measurable Selection Valued Correspondences Induced by Upper Caratheodory Correspondences." *Journal of Fixed Point Theory and Applications* 25, 2. https://doi.org/10.1007/s11784-022-01000-4

[10] Gorniewicz, L., Granas, A., and Kryszewski, W. (1991) "On the Homotopy Method in the Fixed Point Index Theory of Multi-Valued Mappings of Compact Absolute Neighborhood Retracts," *Journal of Mathematical Analysis and Applications* 161, 457-473.

[11] Hildenbrand, W. (1974) *Core and Equilibria of a Large Economy*, Princeton University Press, Princeton.

[12] Himmelberg, C. J. (1975) "Measurable Relations," *Fundamenta Mathematicae* 87, 53-72.

[13] Himmelberg, C. J., and VanVleck, F. S. (1975) "Multifunctions with Values in a Space of Probability Measures," *Journal of Mathematical Analysis and Applications* 50, 108-112.

[14] Hola, L. and Holy, D. (2015) "Minimal USCO and Minimal CUSCO Maps," *Khayyam Journal of Mathematics* 1, 125-150.

[15] Komlos, J. (1967) "A Generalization of a Problem of Steinhaus,"*Acta Mathematica Academiae Scientiarum Hungaricae* 18, 217-229.

[16] Kuratowski, K., and Ryll-Nardzewski, C. (1965) "A General Theorem on Selectors," *Bull. Acad. Polon. Sci. S6r. Sci. Math. Astronom. Phys.*, 13, 397-403.

[17] Levy, Y. (2013) "A Discounted Stochastic Game with No Stationary Nash Equilibrium: Two Examples," *Econometrica* 81, 1973-2007.

[18] Levy, Y., and McLennan, A. (2015) "Corrigendum to: "Discounted Stochastic Games with No Stationary Nash Equilibrium: Two Examples," *Econometrica* 83, 1237-1252.

[19] Lyapunov, A. (1940) "Sur les Fonctions-Vecteurs Compltement Additives," Bulletin of the Academy of Sciences, USSR Series. Mathematics, 4, pp. 465-478.

[20] Mariconda, C. (1992) "Contractibility and Fixed Point Property: The Case of Decomposable Sets," *Nonlinear Analysis: Theory, Methods, and Applications* 18, 689-695.

[21] Page, F. (1991) "Komlos Limits and Fatou's Lemma in Several Dimensions," *Canadian Mathematical Bulletin* 34, 109-112.

[22] Page, F. (2016) "Stationary Markov Equilibria for Approximable Discounted Stochastic Games," Discussion Paper 60, Systemic Risk Centre, London School of Economics, August, 2016.

[23] Pales, Z. and Zeidan, V. (1999) "Characterization of L^1-Closed Decomposable Sets in L^∞," *Journal of Mathematical Analysis and Applications* 238, 491-515.